«Graded automorphisms of the algebra of polynomials in three variables»

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Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]$.

Definition

An automorphism of the algebra $\mathcal{A}$ is elementary if it has the following form

$$\varphi = (x_1, \ldots, x_{i-1},\ x_i + F,\ x_{i+1}, \ldots, x_n),$$

where $F \in \mathbb{K}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$.

Definition

An affine automorphism of the algebra $\mathcal{A}$ is an automorphism φ of the form $x_i \mapsto \sum_{j=1}^{n} c_{ij}x_j + b_i$. If all $b_i = 0$, we call φ linear.

Introduction

We call an automorphism φ *tame* if it is the composition of elementary and linear automorphisms. An automorphism is said to be *wild* if it is not tame. The problem of existence of wild automorphisms is now solved for spaces of dimension less than four:

- For algebra $\mathbb{K}[x]$ automorphism group consists of affine automorphisms: $x \mapsto \lambda x + c$. All of them are tame.
- In 1942 Jung proved that the algebra $\mathbb{K}[x, y]$ admits no wild automorphisms.
- In 1972 Nagata constructed the following automorphism σ of $\mathbb{K}[x, y, z]$

$$\begin{cases} \sigma(x) = x - 2(y^2 + xz)x - (y^2 + xz)^2 z; \\ \sigma(y) = y + (y^2 + xz)z; \\ \sigma(z) = z. \end{cases}$$

In 2004 Shestakov and Umirbaev proved that the Nagata automorphism gives an example of a wild automorphism for $n = 3$.

Notice, that Nagata automorphism preserves following grading:
$(\deg(x), \deg(y), deg(z)) = (3, 1, -1)$.

Graded-wild automorphisms

Let G be a commutative group. Then a *G-grading Γ* of the algebra $\mathcal{A}$ is a decomposition $\mathcal{A}$ into a direct sum of linear subspaces $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$, such that $\mathcal{A}_g \mathcal{A}_h \subset \mathcal{A}_{gh}$. Subspaces $\mathcal{A}_g$ are called *homogeneous components* and all elements of $\mathcal{A}_g$ are called *homogeneous*. If f belongs to $\mathcal{A}_g$ we denote $\deg_\Gamma(f) = g$. If $G = \mathbb{Z}$, for an arbitrary $f = \sum_g f_g, f_g \in \mathcal{A}_g$ we denote by $\deg_\Gamma(f)$ the maximum g such that $f_g \neq 0$. We say that an automorphism φ of the algebra $\mathcal{A}$ respects G-grading Γ if for all $g \in G$ the image of the subspace $\mathcal{A}_g$ under φ is contained in $\mathcal{A}_g$. Such an automorphism is called *graded*.

Definition

If a graded automorphism φ can be decomposed to a composition of graded elementary and linear automorphisms then φ is called graded-tame. Other graded automorphisms are called graded-wild.

Here we will consider the case of homogeneous coordinates.

Nagata automorphism and Anick's automorphism

Nagata automorphism is wild so it is graded-wild.
The known candidate to be nontame in $\mathrm{Aut}(\mathbb{K}[x_1, x_2, x_3, x_4])$ was the Anick automorphism:

$$\begin{cases} \zeta(y_1) = y_1; \\ \zeta(y_2) = y_2 + y_1(y_1y_4 - y_2y_3); \\ \zeta(y_3) = y_3; \\ \zeta(y_4) = y_4 + y_3(y_1y_4 - y_2y_3). \end{cases}$$

In 2008 Arzhantsev and Gaifullin proved that Anick's automorphism is graded-wild with grading $(\deg(y_1), \deg(y_2), \deg(y_3), \deg(y_4)) = (1, 1, -1, -1)$.

Two variables and positive gradings

Proposition

Let $\widetilde{\Gamma}$ be a grading by an abelian group of the polynomial algebra in two variables. Then all graded automorphisms of $\mathcal{A} = \mathbb{K}[x, y]$ are graded-tame with respect to $\widetilde{\Gamma}$.

Proposition

Let degree of x_k be a positive integer for all k. Then all graded automorphisms of the algebra $\mathbb{K}[x_1, \dots, x_n]$ are graded-tame.

Proof.

Without loss of generality, we can assume that $\deg x_1 \leqslant \dots \leqslant \deg x_n$.
Let $\deg x_{n_{k-1}} < \deg x_{n_{k-1}+1} = \dots = \deg x_{n_k} < \deg x_{n_k+1}$. Then

$$x_i \mapsto L_i(x_{n_{k-1}+1}, \dots, x_{n_k}) + f_i(x_1, \dots, x_{n_{k-1}}), n_{k-1} < i \leqslant n_k$$

□

Three variables

Let us consider $\mathbb{Z}$-grading of $\mathbb{K}[x,y,z]$.

Remark

In 1995 Koras and Russel solved linearization problem for 3 variables and 1-dimensional torus, so we can assume coordinates homogeneous in any case.

If we divide all $\deg_\Gamma x$, $\deg_\Gamma y$ and $\deg_\Gamma z$ by their greatest common divisor, then we obtain a new $\mathbb{Z}$-grading which admits a graded-wild automorphism if and only if Γ does. Therefore, we can assume $(\deg_\Gamma(x), \deg_\Gamma(y), \deg_\Gamma(z)) = (a, b, -c)$, where $a, b, c \geq 0$, $a \geq b$ and $\gcd(a, b, c) = 1$ or $(a, b, -c) = (0, 0, 0)$.

Theorem (Trushin)

The grading Γ admits graded-wild automorphisms if and only if one of the following occures

- $(a, b, c) = (0, 0, 0)$.
- a, b, c *strictly positive,* $a = qb + pc$ *for some integers* $p \geq 1$ *and* $q \geq 2$.

Three variables

Let us assume $(\deg_\Gamma(x), \deg_\Gamma(y), \deg_\Gamma(z)) = (a, b, -c)$, where $a, b, c > 0$, $a \geq b$ and $\gcd(a, b, c) = 1$ and let us consider two subgroups of $\mathrm{Aut}_\Gamma(\mathbb{K}[x, y, z])$:

$$\mathrm{E} = \left\{\varphi \in \mathrm{Aut}_\Gamma(\mathbb{K}[x, y, z]) \middle| \varphi(z) = z\right\}, \qquad \mathrm{T} = \{(x, y, \lambda z) \mid \lambda \neq 0\}.$$

Then

Lemma

The group $\mathrm{Aut}_\Gamma(\mathbb{K}[x, y, z])$ *is isomorphic to the semidirect product* $\mathrm{E} \leftthreetimes \mathrm{T}$.

Three variables

The plane $\mathrm{Y} = \mathbb{V}(z-1)$ is invariant under automorphisms from the subgroup E. Therefore, there exists a homomorphism $\alpha : \mathrm{E} \to \mathrm{Aut}(\mathbb{K}[u,v])$, where $u = x\big|_{\mathrm{Y}};\ v = y\big|_{\mathrm{Y}}$, given by formula $\alpha(f,g,z) = (f(u,v,1), g(u,v,1))$. Since f and g homogeneous, α is injective.

Consider the grading $\widetilde{\Gamma}$ of $\mathbb{K}[u,v]$ by the cyclic group $\mathbb{Z}_c$ of order c such that $(\deg_{\widetilde{\Gamma}}(u), \deg_{\widetilde{\Gamma}}(v)) = (\overline{a}, \overline{b})$, where $\overline{a}$ and $\overline{b}$ are the images of a and b under the natural homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_c$.

Remark

Let $\varphi \in \mathrm{E}$. Then the automorphism $\widetilde{\varphi} = \alpha(\varphi) \in \mathrm{Aut}(\mathbb{K}[u,v])$ preserves the grading $\widetilde{\Gamma}$.

We say that the automorphism $\widetilde{\varphi}$ of $\mathbb{K}[u,v]$ can be lifted to an automorphism of $\mathcal{A}$ if the preimage $\alpha^{-1}(\widetilde{\varphi})$ is not empty.

Three variables

Let $\widetilde{\varphi} = (\widetilde{f}, \widetilde{g}) \in \mathrm{Aut}_{\widetilde{\Gamma}}(\mathbb{K}[u,v])$. Then $\alpha^{-1}(\widetilde{\varphi}) = \varnothing$ if and only if the polynomial $\widetilde{f}$ contains a monomial v^q such that $bq < a$ or the polynomial $\widetilde{g}$ has nonzero free term.

Remark

We can assume that $\gcd(b,c) = 1$.

Let us put $\widehat{q} = \max\left\{q \in \mathbb{Z} \middle| bq \equiv a (\mathrm{mod}\ c), bq < a\right\}$
$\left(= \max\left\{q \in \mathbb{Z} \middle| \alpha^{-1}(u + v^q, v) = \varnothing\right\}\right)$.

Example

Let $(\deg(x), \deg(y), deg(z)) = (3, 1, -1)$, then the automorphism $(u + v^2, v)$ can not be lifted and $\widehat{q} = 2$.

Three variables

Lemma

Let $\varphi \in \mathrm{E}$ *is graded-tame automorphism and let* $\widetilde{\varphi} = \alpha(\varphi)$. *Then* $\widetilde{\varphi}(u) = \lambda u + G$, *where* $G \in (u, v)^{\widehat{q}+c}$.

Proposition

If $\widehat{q} \geq 2$, *then the grading* Γ *admits a graded-wild automorphism of the algebra* $\mathbb{K}[x, y, z]$.

Proof.

Let $\tau = (u + v^{\widehat{q}}, v)$ and let $\phi = (u, v + u^{\widehat{l}})$, where $\widehat{l}$ is the smallest possible. Then

$$\tau^{-1} \circ \phi \circ \tau(u) = u + v^{\widehat{q}} - (v + (u + v^{\widehat{q}})^{\widehat{l}})^{\widehat{q}} = u - \widehat{q} v^{\widehat{q}-1} u^{\widehat{l}} + F,$$

where the polynomial $F \in I^{\widehat{l}+\widehat{q}}$. So automorphism $\tau^{-1} \circ \phi \circ \tau$ is wild. □

Example

Let $\tau = \begin{pmatrix} u + v^2 \\ v \end{pmatrix} \in W$ and let $\theta = \begin{pmatrix} u \\ v + u \end{pmatrix} \in D$. Then we have:

$$\tau^{-1} \circ \theta \circ \tau = \begin{pmatrix} u - u^2 - v^4 - 2uv - 2v^3 - 2uv^2 \\ v + u + v^2 \end{pmatrix}$$

Corresponding automorphism of the polynomial algebra in three variables is Nagata automorphism:

$$\sigma = \alpha^{-1}(\tau^{-1} \circ \theta \circ \tau) = \begin{pmatrix} x - x^2z^3 - y^4z - 2xyz - 2y^3 - 2xy^2z^2 \\ y + xz^2 + y^2z \\ z \end{pmatrix}$$

Generator system

Denote the degree of the smallest monomial in a polynomial in one variable f as $\underline{\deg}(f)$. Consider the following sets of $\widetilde{\Gamma}$-graded automorphisms of the algebra $\mathbb{K}[u, v]$:

$$D = \{(u, \lambda v + u^k) | ka \equiv b (\mathrm{mod}\ c), \lambda \in \mathbb{K}^\times\},$$

$$U = \{(\lambda u + f(v), v) | \deg_{\widetilde{\Gamma}}(f(v)) = \overline{a}, \underline{\deg}(f) > \widehat{q}, \lambda \in \mathbb{K}^\times\},$$

$$W = \{(\lambda u + f(v), v) | \deg_{\widetilde{\Gamma}}(f(v)) = \overline{a}, \deg(f) \leq \widehat{q}, \lambda \in \mathbb{K}^\times\}.$$

Lemma

For elementary automorphisms $\tau \in W$, $\theta \in D$, there exists an automorphism $s_{\tau,\theta} \in W$ such that the automorphism $s_{\tau,\theta} \circ \tau^{-1} \circ \theta \circ \tau$ lifts to a space automorphism.

Generator system

Let us introduce the following notation:
$\tau_\theta = s_{\tau,\theta} \circ \tau^{-1}$;
$S = \{\tau_\theta \circ \theta \circ \tau | \tau \in W, \theta \in D\}$.

Lemma

The group $\widetilde{E} = \alpha(E)$ *is generated by the subgroup* U *and the set* S.

Theorem (Trushin)

Automorphisms of the algebra $\mathrm{Aut}_\Gamma(\mathbb{K}[x, y, z])$ *are generated by group* $\alpha^{-1}(U)$, *automorphisms of the form* $\alpha^{-1}(\tau_\theta \circ \theta \circ \tau)$, *where* $\tau \in W, \theta \in D$ *and group* $\mathrm{T} = \{(x, y, \lambda z) \mid \lambda \neq 0\}$.

Example

Let us present the general form of an automorphism of the form $\tau_\theta \circ \theta \circ \tau$.

Let $\tau = \begin{pmatrix} \lambda_1 u + \nu v^2 \\ v \end{pmatrix} \in W$ and let $\theta = \begin{pmatrix} u \\ \lambda_2 v + \mu u^k \end{pmatrix} \in D$. Then

$$\tau^{-1} \circ \theta \circ \tau = \begin{pmatrix} u + \frac{\nu}{\lambda_1} v^2 - \frac{\nu}{\lambda_1} \left(\lambda_2 v + \mu(\lambda_1 u + \nu v^2)^k\right)^2 \\ \lambda_2 v + \mu(\lambda_1 u + \nu v^2)^k \end{pmatrix}$$

The coefficient of v^2 in the polynomial $\tau^{-1} \circ \theta \circ \tau(u)$ is equal to $\dfrac{(1-\lambda_1^2)\nu}{\lambda_1}$.

Thus, the automorphism $\tau_\theta \circ \theta \circ \tau$ is equal to $s_{\tau,\theta} \circ \tau^{-1} \circ \theta \circ \tau$, where

$$s_{\tau,\theta} = \begin{pmatrix} u - \frac{(1-\lambda_1^2)\nu}{\lambda_1 \lambda_2^2} v^2 \\ v \end{pmatrix}.$$

Jung's theorem proof

Let $\varphi = (f, g)$ be an automorphism of the algebra $\mathcal{A} = \mathbb{K}[x, y]$. Consider the mapping $\delta_f : \mathcal{A} \to \mathcal{A}$ such that $\delta_f(h) = J(f, h)$. This map is linear and satisfies the Leibniz rule, so it is a derivation, moreover it is LND. Let us denote by $\widehat{f}$ the top homogeneous component of f with respect to $\mathbb{Z}$-grading Γ. Then $\widehat{f} = Cx^\alpha y^\beta \prod_j (y^q - \lambda_j x^p)$. Notice, that $\delta_{\widehat{f}}$ is also an LND.

Kernel of $\delta_{\widehat{f}}$ is factorially closed and $\delta_{\widehat{f}}(\widehat{f}) = J(\widehat{f}, \widehat{f}) = 0$ that is $\widehat{f} \in \ker \delta_{\widehat{f}}$. Transcendence degree of $\ker \delta_{\widehat{f}}$ over $\mathbb{K}$ is equal to 1, so $\widehat{f} = C(y^q - \lambda x^p)^k, \lambda \neq 0$. We have $\overline{\delta}(x) = -qy^{q-1}$, $\overline{\delta}(y) = -\lambda p x^{p-1}$. If $p > 1$ and $q > 1$, then x divides $\partial(y)$ and y divides $\partial(x)$ so $\partial(x) = 0$ or $\partial(y) = 0$. Since the transcendence degree of $\ker \delta_{\widehat{f}}$ over $\mathbb{K}$ is equal to 1, at least one of p and q is equal to 1. If $p = 1$, let us consider the automorphism $\psi = (x + \frac{1}{\lambda} y^q, y)$. Then $\varphi \circ \psi(x) = s$, where $s = f(x + \frac{1}{\lambda} y^q, y)$. Since this automorphism is homogeneous with respect to the grading Γ, one of the vertices is deleted and the area of coordinate f became smaller.

Two variables graded case proof

Let $\varphi = (f, g)$ be a $\widetilde{\Gamma}$-graded automorphism of $\mathcal{A}$. In the proof of Jung's Theorem we consider a grading Γ of $\mathcal{A}$ and we show, that the top homogeneous component of f with respect to Γ is $\widehat{f} = C(y^q - \lambda x)^k$, $\lambda \neq 0$. Also we show that the automorphism Let us prove that $\psi = (x + \frac{1}{\lambda} y^q, y)$ is also graded with respcet to $\widetilde{\Gamma}$. Since φ is graded, f is homogeneous. So all monomials of $\widehat{f}$ has the same degree. But

$$\widehat{f} = C(y^q - \lambda x)^k = C(y^{qk} - k\lambda x y^{q(k-1)} + \ldots).$$

So $\deg y^{qk} = \deg x y^{q(k-1)}$. Hence, $\deg x = q \deg y$. Therefore, ψ is graded. We prove following

Theorem

Let $\widetilde{\Gamma}$ be a grading by an abelian group of the polynomial algebra in two variables. Then all graded automorphisms of $\mathcal{A} = \mathbb{K}[x, y]$ are graded-tame with respect to $\widetilde{\Gamma}$.

Zeros in grading

If $c = 0, a > b$, then $z \mapsto \lambda z, y \mapsto \mu y$, and hence all automorphisms are tame. If $a = b, c = 0$, then any automorphism of φ has the form $\varphi = (A(z)x + B(z)y,\ C(z)x + D(z)y,\ \kappa z + \mu)$, and the matrix $\Lambda = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$ is non-degenerate for all z, so $\det \Lambda = A(z)D(z) - C(z)D(z) = \lambda \in \mathbb{K}^{\times}$. Hence the greatest common divisor $A(z)$ and $C(z)$ in the ring $\mathbb{K}[z]$ is equal to one, so we can apply Euclid's algorithm to $A(z)$ and $C(z)$.

Now, we reduce φ to the form $\varphi = (A(z)x + B(z)y,\ C(z)x + D(z)y,\ z)$ and then by composition of elementary automorphisms of the form $(x - yq_k, y, z), (x, y - xq_k, z)$ we get that φ has the form $(x + \widetilde{B}(z)y, \widetilde{C}y, z)$. It is clear that this automorphism decomposes into a composition of elementary automorphisms.

Zeros in grading

Now, if $b = 0, a \neq 0, c \neq 0$, then $x \mapsto \lambda_1 x, z \mapsto \lambda_2 z$, hence all graded automorphisms are graded-tame.
Now let $a \neq 0, b = c = 0$. Then $x \mapsto \lambda x$; the images of the variables y and z for any automorphism do not depend on x, and hence, by Jung Theorem, all graded automorphisms are graded-tame.
In the case of strictly positive and strictly negative gradings, all graded automorphisms are graded-tame. So

Lemma

Suppose at least one number among a, b and c equals zero. Let $(a, b, c) \neq 0$. Then there are no graded-wild automorphisms.

- I. Shestakov and U. Umirbaev, The tame and the wild automorphisms of polynomial rings in three variables,
  J. Amer. Math. Soc. 17 (2004), 197–227.
- I.V. Arzhantsev and S.A. Gaifullin, *Cox rings, semigroups and automorphisms of affine algebraic varieties*, Sbornik: Mathematics, **201**:1 (2010), 1–21.
- H. W. E. Jung, Über ganze birationale transformationen der Ebene, J. reine, angew. Math. 184 (1942) 161-174.
- A. Trushin, Gradings allowing wild automorphisms,
  Journal of Algebra and Its Applications 2022 (2021).
- A. Trushin, Graded automorphisms of the algebra of polynomials in three variables, 2023.