On isotropy subgroups

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Workshop : Affine Spaces, Algebraic Group Actions and LNDs

Introduction

First, we will fix some notations.

- k : an alg. closed field of char. 0
- B : an affine k-domain
- $\mathrm{Aut}(B)$: set of k-algebra automorphisms on B.
- $\mathrm{LND}(B)$: set of locally nilpotent derivations on B.
- $\mathrm{Ker}(\delta)$: kernel of LND δ.

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- $\mathrm{Ker}(\delta)$: kernel of LND δ.

There is a natural action of $\mathrm{Aut}(B)$ on $\mathrm{LND}(B)$ defined by $\alpha \cdot \delta = \alpha\delta\alpha^{-1}$, for $\alpha \in \mathrm{Aut}(B)$ and $\delta \in \mathrm{LND}(B)$.

Given $\delta \in \mathrm{LND}(B)$, the **stabilizer** of δ under the above action, i.e., the subgroup $\{\sigma \in \mathrm{Aut}(B) : \sigma\delta = \delta\sigma\}$ of $\mathrm{Aut}(B)$, is called the isotropy subgroup of B with respect to δ and will be denoted by

$$\mathrm{Aut}(B)_\delta \quad \text{or} \quad \mathrm{Aut}(\delta).$$

The big unipotent subgroup $\mathbb{U}(\delta)$

- Every $\delta \in \mathrm{LND}(B)$ induces an element of $\mathrm{Aut}(B)$ via the exponential map, defined as $\exp(\delta) := \sum_{i \geqslant 0} \frac{1}{i!}\delta^i$.
- For any LND δ, each of its replicas $f\delta$ $\big(f \in \mathrm{Ker}(\delta)\big)$ is also an LND.
- Exponents of all replicas of δ form a commutative subgroup $\mathbb{U}(\delta) := \{\exp(f\delta) \mid f \in \mathrm{Ker}(\delta)\}$, called the big unipotent subgroup corresponding to δ.
- The correspondence $f \leftrightarrow \exp(f\delta)$ induces an isomorphism between $\mathbb{U}(\delta)$ and $\big(\mathrm{Ker}(\delta), +\big)$. It is easy to see that for any $\delta \in \mathrm{LND}(B)$, the big unipotent group $\mathbb{U}(\delta)$ is a subgroup of $\mathrm{Aut}(\delta)$.

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- The correspondence $f \leftrightarrow \exp(f\delta)$ induces an isomorphism between $\mathbb{U}(\delta)$ and $\big(\mathrm{Ker}(\delta), +\big)$. It is easy to see that for any $\delta \in \mathrm{LND}(B)$, the big unipotent group $\mathbb{U}(\delta)$ is a subgroup of $\mathrm{Aut}(\delta)$.

Question : Is $\mathbb{U}(\delta) \subsetneq \mathrm{Aut}(\delta)$?

- **Yes**, if when B admits an LND δ' which commutes with δ and $\mathrm{Ker}(\delta) \neq \mathrm{Ker}(\delta')$ (for example, $B = k[X, Y]$, $\delta = \frac{\partial}{\partial X}$ and $\delta' = \frac{\partial}{\partial Y}$). Indeed, if such a δ' exists, then, for any $f \in \mathrm{Ker}(\delta) \cap \mathrm{Ker}(\delta')$, $\exp(f\delta')$ is an element of $\mathrm{Aut}(\delta) \setminus \mathbb{U}(\delta)$.

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  Converse is not true. Take $B = k[X, Y, Z]$ and $\delta = X\frac{\partial}{\partial Y} + 2Y\frac{\partial}{\partial Z}$. Then $A = k[X, XZ - Y^2]$. Define $\phi \in \mathrm{Aut}B$ by $\phi(X) = 2X$, $\phi(Y) = Y$ and $\phi(Z) = Z/2$. Then $\phi|_A \in \mathrm{Aut}(A)$. But $\phi\delta(Z) = 2Y \neq Y = \delta\phi(Z)$. So $\phi \notin \mathrm{Aut}(\delta)$.

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- Let $\delta_1\delta_2 = \delta_2\delta_1$
  Then $f \in \mathrm{Ker}(\delta_1) \cap \mathrm{Ker}(\delta_2) \Rightarrow \exp(f\delta_1) \in \mathrm{Aut}(\delta_2)$.

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  Then $f \in \mathrm{Ker}(\delta_1) \cap \mathrm{Ker}(\delta_2) \quad \Rightarrow \quad \exp(f\delta_1) \in \mathrm{Aut}(\delta_2)$.

- Let $H_f := \{\theta \in \mathrm{Aut}(A) \mid \theta(f) = f\}$. Then

$$C_{\mathrm{Aut}(\delta)}(\exp(f\delta)) = H_f \quad \text{and}$$

$$C_{\mathrm{Aut}(\delta)}(\mathbb{U}(\delta)) = \{\theta \in \mathrm{Aut}(\delta) \mid \theta|_A = id_A\}.$$

Almost rigid domains

Question : What happens when all LNDs are replicas of a canonical one ?

- An affine k-domain B is said to be almost rigid if there exists $D \in \mathrm{LND}(B)$ such that every $\delta \in \mathrm{LND}(B)$ can be written as $\delta = hD$, for some $h \in \mathrm{Ker}(D)$. Moreover, D is called the canonical LND on B.
- For an almost rigid domain B with $B^* = k^*$, if D is a canonical LND and $\phi \in \mathrm{Aut}(B)$, then $\phi D \phi^{-1} = \lambda D$, for some $\lambda \in k^*$.

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Example : $B := \dfrac{k[X, Y, Z]}{(f(X)Y - P(Z))}$, where $\deg_X f > 1$. Then

- (**Bianchi-Veloso (2017)**)[BiV]
  B is almost rigid with the canonical LND D given by $D(x) = 0,\ D(y) = \frac{d\phi}{dz}$ and $D(z) = f(x)$.
- (**Baltzar-Veloso (2021)**)[BaV]
  Let $\delta \in \mathrm{LND}(B)$. Then $\mathrm{Aut}(\delta)$ is generated by a finite cyclic group of the form

$$\{(\lambda x, y, z) \mid \lambda \in k^* \text{ and } \lambda^s = 1\} \text{ and } \mathbb{U}(D),$$

where $f(X) = X^j h(X^s)$ such that $h \in k^{[1]}$ has a non-zero root.

Generalised Danielewski surfaces

Definition.

A k-algebra B is said to be a generalised Danielewski surface over k if B is isomorphic to the k-algebra

$$B_{d,P} := \frac{k[X, Y_1, Y_2]}{(X^d Y_2 - P(X, Y_1))},$$

where $d \geqslant 2$ and $r := \deg_{Y_1}(P) \geqslant 2$. If $P(X, Y_1) = \prod_{i=1}^{r} \big(Y_1 - \sigma_i(X)\big)$, where $\sigma_i(X) \in k[X]$, then the surface $B_{d,P}$ is called a generalised Danielewski surface in standard form.

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$B_{d,P}$ is an almost rigid domain with the canonical lnd $D_{d,P}$, given by

$$D_{d,P} := x^d \frac{\partial}{\partial y_1} + \frac{\partial P}{\partial y_1} \frac{\partial}{\partial y_2}.$$

The automorphism group of $B_{d,P}$ was studied by **A. Dubouloz** and **P-M. Poloni** ([DP], **2009**).

$\mathrm{Aut}(B_{d,P})$

$\mathcal{S}_r$: the symmetric group of r elements. id : the identity permutation

- Every automorphism Φ in $\mathrm{Aut}(B_{d,P})$ is uniquely determined by the datum $\mathcal{A}_\Phi = (\alpha, \mu, a, b(x)) \in \mathcal{S}_r \times k^* \times k^* \times k[x]$, such that the polynomial $c(x) := \sigma_{\alpha(i)}(ax) - \mu\sigma_i(x)$ does not depend on the index $i = 1, 2, \ldots, r$.
- Φ is induced by $\Psi \in \mathrm{Aut}(k[X, Y_1, Y_2])$ given by

$$X \to aX, \quad Y_1 \to \mu Y_1 + \tilde{c}(X) \text{ and}$$

$$Y_2 \to \frac{1}{a^d}\mu^r Y_2 + \frac{1}{(aX)^d}\Big(\prod_{i=1}^{r}(\mu Y_1 + \tilde{c}(X) - \sigma_i(aX)) - \mu^r P(X, Y_1)\Big),$$

where $\tilde{c}(X) = c(X) + X^d b(X)$.

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  where $\tilde{c}(X) = c(X) + X^d b(X)$.
- The composition $\Phi_2 \circ \Phi_1$ of two automorphisms Φ_1 and Φ_2 of $B_{d,P}$ with data $\mathcal{A}_{\Phi_1} = (\alpha_1, \mu_1, a_1, b_1(x))$ and $\mathcal{A}_{\Phi_2} = (\alpha_2, \mu_2, a_2, b_2(x))$ respectively is the automorphism of $B_{d,P}$ with datum $\mathcal{A} = \big(\alpha_2\alpha_1, \mu_2\mu_1, a_2 a_1, \frac{1}{a_2^d}\mu_2 b_1(x) + b_2(a_1 x)\big)$.

$\mathrm{Aut}(B_{d,P})$

- Let $\mathbb{U}, \mathbb{H}, \mathbb{S}$ be the subgroups of $\mathrm{Aut}(B_{d,P})$ consisting of the automorphisms corresponding to the data of the type $(id, 1, 1, b(x)), (id, 1, a, 0)$ and $(\alpha, \mu, 1, 0)$ respectively.

- $\mathrm{Aut}(B_{d,P}) \cong (\mathbb{S} \times \mathbb{H}) \ltimes \mathbb{U}$.

$\mathrm{Aut}(D_{d,P})$

Lemma

Let $\delta \in \mathrm{LND}(B_{d,P})$ and $\alpha \in \mathrm{Aut}(B_{d,P})$. Then the following are equivalent.

- $\alpha \in \mathrm{Aut}(\delta)$.
- $\delta\alpha(y_1) = \alpha\delta(y_1)$.
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For $G \leqslant \mathrm{Aut}(B_{d,P})$, define $G_\delta := G \cap \mathrm{Aut}(\delta)$.

Theorem I(-,Lahiri)

Let $\delta = f(x)D_{d,P}$, where $f(x) = \sum_{i=0}^{l} a_i x^{n_i} \in k[x]$ ($n_i, l \in \mathbb{N} \cup \{0\}, a_i \in k^*$ for each i). Suppose $\mathbb{G} := \mathrm{Aut}(\delta)$ and $n := \mathrm{GCD}(d+n_0, \ldots, d+n_l)$. Then the following statements hold.

- The unipotent group $\mathbb{U} \subseteq \mathbb{G}$ and hence $\mathbb{U}_\delta = \mathbb{U}$.
- If $a \in k^*$ with $a^q \neq 1$ for any $q \in \{1, \ldots, d-1\}$, then $\big(id, 1, a, 0\big) \in \mathbb{G}$ if and only if $a^n = 1$ and $n \geqslant d$.

$\mathrm{Aut}(D_{d,P})$

Theorem

- If $a \in k^*$ with $a^{q_0} = 1$ for some minimal $q_0 \in \{2, \ldots, d-1\}$, then $(id, 1, a, 0) \in \mathbb{G}$ if and only if $q_0 \mid n$.
- $\mathbb{H}_\delta \cong \mathbb{Z}_n$.
- The subgroup $\mathbb{S}_\delta = \{Id_{B_{d,P}}\}$.
- The isotropy subgroup $\mathbb{G} \cong (\mathbb{H} \times \mathbb{S})_\delta \ltimes \mathbb{U}$.

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- The isotropy subgroup $\mathbb{G} \cong (\mathbb{H} \times \mathbb{S})_\delta \ltimes \mathbb{U}$.

Remark

- If $\delta = D_{d,P}$, then $\mathbb{U}(\delta) \subsetneq \mathrm{Aut}(\delta)$. Indeed, if $\omega \in k$ be a primitive d-th root of unity, then $\big(id, 1, \omega, 0\big)$ $(\neq Id_{B_{d,P}}) \in \mathrm{Aut}(\delta)$.
- However, it may happen that $\mathrm{Aut}(\delta) = \mathbb{U}$, when δ is a replica of $D_{d,P}$. For example, if $\delta = (x + x^2)D_{d,P}$, then $(\mathbb{H} \times \mathbb{S})_\delta = \{Id_{B_{d,P}}\}$.

Danielewski varieties with constant coefficients

Definition

An affine algebraic variety $\mathbb{V}_{\mathrm{con}} \subseteq \mathbb{K}^{m+1}$ is called a Danielewski variety with constant coefficients if

$$\mathbb{K}[\mathbb{V}_{\mathrm{con}}] = \frac{\mathbb{K}[Y_1, Y_2, \ldots, Y_m, Z]}{(Y_1 Y_2^{k_2} \ldots Y_m^{k_m} - P(Z))},$$

where $m, k_1, \ldots, k_m, \deg_Z(P) \geqslant 2$, P monic.

These varieties were introduced by **Dubouloz (2007)** [D07] as counterexamples to the Generalized Zariski Cancellation Problem.

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Theorem (Dubouloz)

If $\mathbb{K}[\mathbb{V}_{\mathrm{con}}] = \dfrac{\mathbb{K}[Y_1, Y_2, \ldots, Y_m, Z]}{(Y_1 Y_2^{k_2} \ldots Y_m^{k_m} - P(Z))}$ and $\mathbb{K}[\mathbb{V}'_{\mathrm{con}}] = \dfrac{\mathbb{K}[Y_1, Y_2, \ldots, Y_m, Z]}{(Y_1 Y_2^{k'_2} \ldots Y_m^{k'_m} - P(Z))}$, such that $(k_2, \ldots, k_m) \neq (k'_2, \ldots, k'_m)$, then $\mathbb{V}_{\mathrm{con}} \times \mathbb{K} \cong \mathbb{V}'_{\mathrm{con}} \times \mathbb{K}$ but $\mathbb{V}_{\mathrm{con}} \ncong \mathbb{V}'_{\mathrm{con}}$.

Group actions on $\mathbb{V}_{\text{con}}$

Later **Gaifullin (2021)**[G21] studied $\mathrm{Aut}(\mathbb{V}_{\text{con}})$ in detail.

- The stabilizer of the monomial $Y_1 Y_2^{k_2} \dots Y_m^{k_m}$ under the natural diagonal action of the m-dimensional algebraic torus $(\mathbb{K}^*)^m$ on $\mathbb{K}[Y_1, Y_2, \dots, Y_m]$ is isomorphic to the $(m-1)$-dimensional torus, denoted by $\mathbb{T}$. If we consider the trivial action of $\mathbb{T}$ on $\mathbb{K}[Z]$, then there is an effective action of $\mathbb{T}$ on $\mathbb{V}_{\text{con}}$. $\mathbb{T}$ is called the proper torus of $\mathbb{V}_{\text{con}}$.
- There is a natural action of the symmetric group $\mathcal{S}_m$ on $\mathbb{K}[Y_1, \dots, Y_m]$. The stabilizer $\mathbb{S}$ of the monomial $Y_1 Y_2^{k_2} \dots Y_m^{k_m}$ is isomorphic to the group $\mathcal{S}_{m_1} \times \cdots \times \mathcal{S}_{m_n}$, where $m = m_1 + \cdots + m_n$ and for each $i \in \{1, \dots, n\}$, $\mathcal{S}_{m_i}$ permutes the m_i many Y_j's with same k_j. If we consider the trivial action of $\mathbb{S}$ on $\mathbb{K}[Z]$, then there is an effective action of $\mathbb{S}$ on $\mathbb{V}_{\text{con}}$. This group $\mathbb{S}$ is called the symmetric group of $\mathbb{V}_{\text{con}}$.

Different types of actions on $\mathbb{V}_{\rm con}$

- If $P(Z) = Z^d$, there is also an effective action of an one-dimensional torus $\mathbb{K}^*$ acting by

$$t \cdot (y_1, y_2, \dots, y_m, z) = (t^d y_1, y_2 \dots, y_m, tz), \quad \text{for all } t \in \mathbb{K}^*.$$

If $P(Z) \neq Z^d$ and v is the maximal integer such that there exists a polynomial $Q(Z) \in \mathbb{K}[Z]$ and a non-negative integer u such that $P(Z) = Z^u Q(Z^v)$, then there is an action of $\mathbb{Z}_v$ (considered as a subgroup of k^*) on $\mathbb{V}_{\rm con}$, given by

$$t \cdot (y_1, y_2, \dots, y_m, z) = (t^u y_1, y_2 \dots, y_m, tz), \quad \text{where } t^v = 1.$$

In each of these cases, the groups $\mathbb{K}^*$ and $\mathbb{Z}_v$ are called the additional quasitorus of $\mathbb{V}_{\rm con}$ and denoted by $\mathbb{D}$.

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  In each of these cases, the groups $\mathbb{K}^*$ and $\mathbb{Z}_v$ are called the additional quasitorus of $\mathbb{V}_{\mathrm{con}}$ and denoted by $\mathbb{D}$.

- $\mathbb{K}[\mathbb{V}_{\mathrm{con}}]$ is an almost rigid domain with the canonical LND

$$D_{\mathrm{con}} := P'(z)\frac{\partial}{\partial y_1} + y_2^{k_2} \ldots y_m^{k_m} \frac{\partial}{\partial z}$$

  .

- $\mathrm{Aut}(\mathbb{V}_{\mathrm{con}}) \cong \mathbb{S} \ltimes \big((\mathbb{T} \times \mathbb{D}) \ltimes \mathbb{U}(D\mathrm{con})\big).$

Lemma

Let $\delta = hD_{con}$, where $h \in \mathbb{K}[y_2, \ldots, y_m]$ and $\theta \in \mathbb{T}$. Then the following statements are equivalent.

- $\delta\theta(y_1) = \theta\delta(y_1)$.
- $h\theta(y_1) = y_1\theta(h)$.
- $\delta\theta(z) = \theta\delta(z)$.
- $\theta \in \mathrm{Aut}(\delta)$.

Corollary

Let $\delta = hD\mathrm{con}$, for some $h \in \mathbb{K}^*$. Then $\mathbb{T}_\delta \cong (\mathbb{K}^*)^{m-2} \times \mathbb{Z}_s$, where $s = \mathrm{GCD}(k_2, \ldots, k_m)$.

$\mathbb{S}_\delta$ and $\mathbb{D}_\delta$

Lemma

Let $\delta = hD\mathrm{con}$, for some $h \in \mathbb{K}[y_2, \ldots, y_m]$ and $\sigma \in \mathbb{S}$. Then the following statements are equivalent.

- $\delta\sigma(y_1) = \sigma\delta(y_1)$.
- $\sigma(h) = h$.
- $\delta\sigma(z) = \sigma\delta(z)$.
- $\sigma \in \mathrm{Aut}(\delta)$.

In particular, if $h \in \mathbb{K}^*$, then $\mathbb{S}$ $(= \mathbb{S}_\delta)$ is a subgroup of $\mathrm{Aut}(\delta)$.

Lemma

Let $\delta = hD_{\mathrm{con}}$, for some $h \in \mathbb{K}[y_2, \ldots, y_m]$. Let $\varphi \in \mathbb{D}$. Then

$$\varphi \in \mathrm{Aut}(\delta) \Leftrightarrow \varphi = Id.$$

$\mathrm{Aut}(\mathbb{V}_{\mathrm{con}})$ and $\mathrm{Aut}(D_{\mathrm{con}})$

Theorem (Gaifullin)

$\mathrm{Aut}(\mathbb{V}_{\mathrm{con}}) \cong \mathbb{S} \ltimes \big((\mathbb{T} \times \mathbb{D}) \ltimes \mathbb{U}(D_{\mathrm{con}})\big).$

Theorem II(-,Lahiri)

The isotropy subgroup $\mathrm{Aut}(D\mathrm{con})$ can be described as follows.

- If $P(Z) = Z^d$, then

$$\mathrm{Aut}(D_{\mathrm{con}}) \cong \mathbb{S} \ltimes \big((\mathbb{K}^*)^{m-1} \ltimes \mathbb{U}(D_{\mathrm{con}})\big).$$

- If $P(Z) \neq Z^d$ and v is the maximal integer such that $P(Z) = Z^u Q(Z^v)$, then

$$\mathrm{Aut}(D\mathrm{con}) \cong \mathbb{S} \ltimes \Big(\big((\mathbb{K}^*)^{m-2} \times \mathbb{Z}_{sv}\big) \ltimes \mathbb{U}(D\mathrm{con})\Big),$$

where $s = \mathrm{GCD}(k_2, \ldots, k_m)$.

Some Remarks

- If $P(Z) = Z^d$, then there exist $\sigma_1 \in \mathbb{T}$ and $\sigma_2 \in \mathbb{D}$ such that $\sigma_1\sigma_2 \in (\mathbb{T} \times \mathbb{D})_{D\mathrm{con}}$ but neither $\sigma_1 \in \mathbb{T}_{D\mathrm{con}}$ nor $\sigma_2 \in \mathbb{D}_{D\mathrm{con}}$ $(= \{Id\})$. For example, let $\sigma_1(y_1, \ldots, y_m, z) = (\frac{1}{\lambda^{k_2}} y_1, \lambda y_2, y_3, \ldots, y_m, z)$ and $\sigma_2(y_1, y_2, \ldots, y_m, z) = (\lambda^{k_2 d} y_1, y_2, \ldots, y_m, \lambda^{k_2} z)$, where $\lambda \in \mathbb{K}^*$ be such that $\lambda^{k_2} \neq 1$.
- If $P(Z) \neq Z^d$ and $v \geqslant 2$ be the maximal integer such that $P(Z) = Z^u Q(Z^v)$ then there exist $\sigma_1 \in \mathbb{T}$ and $\sigma_2 \in \mathbb{D}$ such that $\sigma_1\sigma_2 \in (\mathbb{T} \times \mathbb{D})_{D\mathrm{con}}$ but neither $\sigma_1 \in \mathbb{T}_{D\mathrm{con}}$ nor $\sigma_2 \in \mathbb{D}_{D\mathrm{con}}$ $(= \{Id\})$. For example, let $\sigma_1(y_1, \ldots, y_m, z) = (\frac{1}{\lambda^{k_2}} y_1, \lambda y_2, y_3, \ldots, y_m, z)$ and $\sigma_2(y_1, y_2, \ldots, y_m, z) = (\lambda^{k_2 u} y_1, y_2, \ldots, y_m, \lambda^{k_2} z)$, where $\lambda \in \mathbb{K}^*$ be such that $\lambda^{k_2 v} = 1$ but $\lambda^{k_2} \neq 1$.

A threefold by Finston and Maubach

Definition

Consider the Pham-Brieskorn surface

$$R = \frac{\mathbb{C}[X, Y, Z]}{(X^a + Y^b + Z^c)}, \quad \text{where } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1.$$

For $m, n \geqslant 2$, we define the following threefold

$$B_{m,n} = \frac{R[U, V]}{(X^m U - Y^n V - 1)}.$$

They were introduced by **Finston** and **Maubach** in **2008** as another set of examples to the Zariski Cancellation Problem.

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They were introduced by **Finston** and **Maubach** in **2008** as another set of examples to the Zariski Cancellation Problem. They showed

- $B_{m,n}$ is a UFD, but not regular.
- If $(m, n) \neq (m', n')$, then $B_{m,n} \ncong B_{m',n'}$ but ${B_{m,n}}^{[1]} \cong {B_{m',n'}}^{[1]}$.
- $B_{m,n}$ is almost rigid with canonical LND

$$D_{m,n} = y^n \frac{\partial}{\partial u} + x^m \frac{\partial}{\partial v}.$$

$\mathrm{Aut}(B_{m,n})$

Theorem (Finston, Maubach)

$\mathrm{Aut}(B_{m,n})$ is generated by the following elements.

- $\theta_f^+(x,y,z,u,v) \to (x,y,z,u+f(x,y,z)y^n, v+f(x,y,z)x^m)$, for $f \in R$.
- $\theta_\mu^*(x,y,z,u,v) \to (\mu^{bc}x, \mu^{ac}y, \mu^{ab}z, \frac{1}{\mu^{mbc}}u, \frac{1}{\mu^{nac}}v)$, for $\mu \in \mathbb{C}^*$.

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So $\mathrm{Aut}(B_{m,n}) \cong \mathbb{C}^* \ltimes \mathbb{U}(D_{m,n})$.

Let $\mathbb{T}$:= set of automorphisms induced by the action of $\mathbb{C}^*$.

Lemma

Let $\delta \in hD_{m,n}$, where $h \in R \setminus \{0\}$ and $\theta := \theta_\mu^* \in \mathbb{T}$. Then TFAE

- $\theta \in \mathrm{Aut}(\delta)$,
- $\delta\theta(\mu) = \theta\delta(\mu)$,
- $\delta\theta(v) = \theta\delta(v)$,
- $\mu^{nac+mbc}\theta(h) = h$.

$\mathrm{Aut}(\delta)$

Theorem (-,Lahiri)

Let $\delta \in hD_{m,n}$, where $h \in R \setminus \{0\}$. Then

- $\mathbb{T}_\delta$ is a finite cyclic group.
- $\mathrm{Aut}(\delta) \cong \mathbb{T}_\delta \rtimes \mathbb{U}(D_{m,n})$.

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Remarks

- If $h = \sum_{r,s,t \geqslant 0, r<a} a_{r,s,t} x^r y^s z^t$, then $\theta_\mu \in \mathrm{Aut}(\delta) \Leftrightarrow \mu^{bc(m+r)+ac(n+s)+abt} = 1$.
- If $\delta = D_{m,n}$, then $\mathbb{U}(\delta) \subsetneqq \mathrm{Aut}(\delta)$ as $mbc + nac > 2$.
- Let $a > 2$ and $\delta = (x + x^2)D_{m,n}$. Then $\mathbb{T}_\delta = \{id_{B_{m,n}}\}$ and hence $\mathbb{U}(\delta) = \mathrm{Aut}(\delta)$.

Isotropy subgroups in $k^{[2]}$

Theorem (Rentschler (1968))[R]

Let $D(\neq 0) \in \mathrm{LND}(k[X, Y])$. There there exists an automorphism $\alpha \in \mathrm{Aut}(k[X, Y])$ and $f(X) \in k[X]$ such that $\alpha D \alpha^{-1} = f(X)\dfrac{\partial}{\partial Y}$.

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Theorem (Baltzar, Veloso (2021))[BaV]

Let $D = f(X)\dfrac{\partial}{\partial Y}$, where $f(X) \in k[X]$. Then all elements of $\mathrm{Aut}(D)$ are of the form

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} aX + b \\ cY + p(X) \end{pmatrix},$$

where $a, b, c \in k,\ bc \neq 0, p(X) \in k[X],\ f(aX + b) = cf(X)$.

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Remarks

- If $n := \deg_X f$, then $c = a^n$.
- $\mathrm{Aut}(D) = \mathbb{U}(\frac{\partial}{\partial Y})$ if and only if $\{\phi|_{k[X]} \mid \phi \in \mathrm{Aut}(D) \text{ and } \phi(f) = \lambda f,\ \lambda \in k^*\} = \{id\}$.

An elementary result

Let $B \in \mathrm{LND}(k^{[3]})$ and $D_1, D_2 \in \mathrm{LND}(B)$.

Proposition

$\mathbb{U}(D_2) \subseteq \mathrm{Aut}(D_1) \Leftrightarrow \mathrm{Ker}(D_1) = \mathrm{Ker}(D_2)$.

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Remarks

- $\mathrm{Aut}(D_1) = \mathrm{Aut}(D_2) \Rightarrow \mathrm{Ker}(D_1) = \mathrm{Ker}(D_2)$.
- Converse need not be true. Let $B = k[X, Y, Z]$, $D_1 = Y\dfrac{\partial}{\partial Z}$ and $D_2 = X\dfrac{\partial}{\partial Z}$. Then $\mathrm{Ker}(D_1) = \mathrm{Ker}(D_2) = k[X, Y]$. Consider the automorphsim ϕ of B given by

$$\phi(X, Y, Z) = (X + Y, Y, Z + X).$$

Then $\phi \in \mathrm{Aut}(D_1) \setminus \mathrm{Aut}(D_2)$.

Triangularizable LNDs

Definition

An LND D on $B = k^{[3]}$ is said to be triangularizable if there exists a coordinate system (X, Y, Z) of B with respect to which D is triangular, i.e.,

$$D(X) = 0, \quad D(Y) \in k[X] \text{ and } D(Z) \in k[X, Y].$$

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Theorem III(-, Gaifullin)

Let $B = k^{[3]}$ and $D(\neq 0) \in \mathrm{LND}(B)$. TFAE

- There exists a locally nilpotent derivation δ of **rank** 1 such that $Exp(\delta) \in \mathrm{Aut}(D)$.
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Remark

Suppose $B = k[X, Y, Z]$ and $\delta = f(X, Y)\dfrac{\partial}{\partial Z}$, then one can get the desired coordinate system by applying a "tame" automorphism.

An Example in $k^{[4]}$

Let $B = k[X, Y, Z]$ and

$$\Delta = X\frac{\partial}{\partial Y} + 2Y\frac{\partial}{\partial Z}.$$

Then

$$\mathrm{Ker}(\Delta) = k[X, F] \ \text{ where } \ F = XZ - Y^2.$$

For each $t \in k^*$, let $D_t := tF\Delta$. Extend D_t to $\tilde{D}_t \in \mathrm{LND}(B[W])$ by setting $\tilde{D}_t(W) = 0$. Then

- D_t is not triangularizable (**Bass (1984)**[B84]).
- $\tilde{D}_t$ is not triangularizable. (**Freudenburg**[F]).
- But $\dfrac{\partial}{\partial W} \in \mathrm{Aut}(\tilde{D}_t)$.

So Theorem III does not extend to higher dimensions.

Rank one LNDs

Definition

Let B be an affine k-domain and $f \in B$. Define

$$\mathrm{Aut}(B)^{(f)} := \{\theta \in \mathrm{Aut}(B) \mid \theta(f) = \lambda f, \text{ for some } \lambda \in k^*\}.$$

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Let B be an affine domain and $B^* = k^*$. Let $D, \delta \in \mathrm{LND}(B)$ such that δ has a slice and $D = h\delta$ for some $h \in \mathrm{Ker}(\delta) := A$. Then

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Corollary I(-, Gaifullin)

Let $B = k^{[3]}$ and $D \in \mathrm{LND}(B)$. Then
$\mathrm{Aut}(D) \cong \mathrm{Aut}(k^{[2]}) \ltimes \mathbb{U}(D)$ in the following cases :

- D is fixed point free.
- $D^2(X) = D^2(Y) = D^2(Z) = 0$.

A useful Lemma

Lemma

Let $B = k^{[3]}$, $D(\neq 0) \in \mathrm{LND}(B)$ and $A = \mathrm{Ker}(D)$. Let g be a local slice of D with $D(g) = f \in A$. If $\delta := D \mid_{A[g]}$, then

- $\delta = f\dfrac{\partial}{\partial g} \in \mathrm{LND}(A[g])$.
- there exists a **injective** group homomorphism $\Phi : \mathrm{Aut}(D) \to \mathrm{Aut}(\delta)$.
- $\mathrm{Aut}(\delta) \cong \mathrm{Aut}(A)^{(f)} \ltimes \mathbb{U}(\dfrac{\partial}{\partial g})$.
- $\Phi(\mathbb{U}(D)) = \{Exp(h\delta) \mid h \in A\}$.

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- $\Phi(\mathbb{U}(D)) = \{Exp(h\delta) \mid h \in A\}$.

Remark

Any $\phi \in \mathrm{Aut}(A)^{(f)}$ with $\phi(f) = \lambda f$ $(\lambda \in k^*)$ can be uniquely extended to an element of $\mathrm{Aut}(A[g])$ by setting $\phi(g) = \lambda g$.

Rank 2 LNDs

Definition

An LND D on the affine domain B is said to be **reducible** if there exists $b \in B$ such that $D(B) \subseteq (b)B$. Otherwise, D is **irreducible**.

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Let $B = k^{[3]}$ and $D \in \mathrm{LND}(B)$ be irreducible of rank at most 2. Then there exists a variable X of B and $g \in B$ such that $D(g) = f(X)$.

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Lemma

Let $B = k[X, Y, Z]$, $D(\neq 0) \in \mathrm{LND}(B)$ be irreducible of rank 2 and $A = \mathrm{Ker}\ D$. Assume that $D(X) = 0$.

(i) There exist $v, g \in B$ such that $A = k[X, v]$ and $D(g) = f(X)$,

(ii) $\mathrm{Aut}(\delta) \cong \mathrm{Aut}(A)^{(f)} \ltimes U(\frac{\partial}{\partial g})$, where $\delta = D|_{A[g]}$ and

Rank 2 LNDs

Lemma

(iii) if $n := \deg_X f$, then $\mathrm{Aut}(A)^{(f)} =$

$$\left\{\phi \in \mathrm{Aut}(A) \;\middle|\; \begin{array}{l} \phi(X) = aX + b \text{ where } a \in k^*,\ b \in k,\ \phi(f) = a^n f, \\ \phi(v) = \mu v + \beta(X),\ \mu \in k^* \text{ and } \beta(X) \in k[X]. \end{array} \right\}$$

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Let us look at the following example due to Bass.
$D(X) = 0, \quad D(Y) = X \quad \text{and} \quad DZ = -2Y$.

- D is triangular of rank 2.
- $\mathrm{Ker}\ D = A = k[X, v]$, where $v = XZ + Y^2$ and $D|_A := \delta = X\dfrac{\partial}{\partial Y}$.
- $A \cap D(B) = (X)$.
- $G \cong \left\{\phi \in \mathrm{Aut}(A) \;\middle|\; \begin{array}{l} \phi(X) = \lambda X, \quad \lambda \in k^*, \\ \phi(v) = \lambda^2 v + X\beta(X), \quad \beta(X) \in k[X]. \end{array} \right\}$
- $\mathrm{Aut}(\delta) \cong G \ltimes \mathbb{U}(D)$.

References

[BaV] R. Baltazar and M. O. Veloso, *On isotropy group of Danielewski surfaces*, Comm. Alg. **49(3)** (2021) 1006–1016.

[B84] H. Bass, *A non-triangular action of $\mathbb{G}_a$ on $\mathbb{A}^3$*, J. Pure Appl. Algebra **33** (1984) 1–5.

[BiV] A. C. Bianchi and M. O. Veloso, *Locally nilpotent derivations and automorphism groups of certain Danielweski surfaces*, J. Algebra **469** (2017) 96–108.

[DL] N. Dasgupta, A. Lahiri, *Isotropy subgroup of some almost rigid domains*, Journal of Pure and Applied Algebra, **227(4)** (2023).

[D07] A. Dubouloz, *Additive group actions on Danielweski varieties and the cancellation problem*, Math. Z. **255(1)** (2007) 77–93.

[DP] A. Dubouloz and P-M. Poloni, *On a class of Danielewski surfaces in affine 3-space*, J. Algebra **321** (2009) 1797–1812.

[F] G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations, Second Edition*, Springer-Verlag, Berlin, Heidelberg (2017).

[G21] S. Gaifullin, *Automorphisms of Danielewski varieties*, J. Algebra **573** (2021) 364–392.

Thank you!