Residual coordinates of affine fibrations and their applications

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Notation

Notation

- $R :=$ A commutative ring with unity.
- $A :=$ An R-algebra.
- $K :=$ Quotient field of R, whenever R is domain.
- $Spec(R) :=$ Collection of all prime ideals of R.
- $k(P) :=$ The residue field R_P/PR_P, where $P \in Spec(R)$.
- $R^{[n]} :=$ The polynomial algebra in n variables over R.
- $A = R^{[n]} :=$ A is isomorphic, as an R-algebra, to the polynomial algebra $R^{[n]}$.
- $\Omega_R(A) :=$ module of differentials of A over R.

Definitions

Definitions

- $D : A \longrightarrow A$ is an R-derivation of A if $\forall\ a, b \in A$ and $r_1, r_2 \in R$, $D(r_1 a + r_2 b) = r_1 D(a) + r_2 D(b)$ and $D(ab) = aD(b) + bD(a)$.

  Let D be an R-derivation of A.

- D is called a locally nilpotent R-derivation ($R-$LND) of A if for each $x \in A$, there exists $n \in \mathbb{N}$ such that $D^n(x) = 0$, i.e, $D^{n-1}(x) \in \mathrm{Ker}(D)$.
- D is said to have a slice $s \in A$ if $D(s) = 1$, i.e., D is surjective.
- D is called fixed point free (FPF) if $D(A)A = A$.

Definitions

- A is said to be an $\mathbb{A}^n$-fibration over R if it is finitely generated and flat over R, and satisfies $A \otimes_R k(P) = k(P)^{[n]}$ for all $P \in Spec(R)$.
- A is called stably polynomial over R if $A^{[m]} = R^{[n]}$ for some $m, n \in \mathbb{N}$.
- Let A be an $\mathbb{A}^n$-fibration over R. $F \in A$ is called a m-stable coordinate of A if $A^{[m]} = R[F]^{[m+n-1]}$.
- R is called a retract of A if there is an R-algebra surjection $\phi : A \longrightarrow R$ and $\phi|_R = id_R$.

  Let $R \subseteq A$ be domains.
- R is called inert or factorially closed in A if $f, g \in A$ and $fg \in R$ implies that $f, g \in R$.

A few well-known results

- **Slice Theorem:** $\mathbb{Q} \hookrightarrow R$ a ring. If D is an R-LND of A with a slice s, then $A = \text{Ker}(D)[s] = \text{Ker}(D)^{[1]}$.
- If A is a domain, then $\text{Ker}(D)$ is inert in A and $\text{tr.deg}_{\text{Ker}(D)}(A) = 1$.
- If $\mathbb{Q} \hookrightarrow R$ is Noetherian and $A^{[m]} = R^{[m+2]}$, then A is an $\mathbb{A}^2$-fibration over R.

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Residual coordinates of $R[X, Y]$ ([Bha88])

To generalized Abhyankar-Moh & Suzuki Epimorphism Theorem

Definition (Bhatwadekar, [Bha88])

$W \in R[X, Y]$ is called a residual coordinate if
$R[X, Y] \otimes_R k(P) = (R[W] \otimes_R k(P))^{[1]} = k(P)[\overline{W}]^{[1]}$ for each $P \in \mathrm{Spec}(R)$.

Example 1

$R = \mathbb{Q}[t]_{(t)}$, $F = t^2Y + (X + tY^2) + t(X + tY^2)^2 \in R[X, Y]$

$\mathrm{Spec}(R) = \{0, tR\}$.

In $R[X, Y] \otimes_R k(0) = \mathbb{Q}(t)[X, Y]$: $\overline{F}$ is a variable of $\mathbb{Q}(t)[X, Y]$
$(X, Y) \longrightarrow (X + tY^2,\ Y) \longrightarrow (X + tY^2,\ t^2Y + (X + tY^2) + t(X + tY^2)^2)$

In $R[X, Y] \otimes_R k(tR) = R/tR[X, Y] = \mathbb{Q}[X, Y]$: $\overline{F} = X$ is a variable of $\mathbb{Q}[X, Y]$.

So, F is a residual variable of $R[X, Y]$. **Question:** Is F a variable of $R[X, Y]$?

Residual coordinates of $R[X, Y]$ ([Bha88])

His observations:

- If F is a residual coord. of $R[X, Y]$, then $R[X, Y] \otimes_{R[F]} k(Q) = k(Q)^{[1]}$ for all $Q \in \mathrm{Spec}(R[F])$.
- If F is a residual coord. of $R[X, Y]$, then $\Omega_{R[F]}(R[X, Y])$ is free of rank one over $R[X, Y]$.
- If R is a domain of characteristics zero and $R[X, Y]/(F) = R^{[1]}$, then F is a residual coord. of $R[X, Y]$.
- If $\dim(R) < \infty$, R is a Noetherian ring and $R[X, Y]/(F) = R^{[1]}$, then $R[X, Y]$ is $R[F]$-flat.

Application: Generalized Epimorphism Theorem

Proving Generalized Epimorphism Theorem

Theorem 2 (Generalized Epimorphism Theorem, [Bha88])

R a ring, either contains $\mathbb{Q}$ or R is a seminormal domain of characteristic zero.

$R[X,Y]/(F) = R^{[1]} \implies R[X,Y] = R[F]^{[1]}$

Sketch of proof:

By proper reduction arguments can assume that
R is a finite dimensional Noetherian domain either contains $\mathbb{Q}$ or is seminormal.

$R[X,Y]/(F) = R^{[1]} \implies$

- F is a residual coord. of $R[X,Y] \implies R[X,Y] \otimes_{R[F]} k(Q) = k(Q)^{[1]}$ for all $Q \in \mathrm{Spec}(R[F])$.
- $R[X,Y]$ is $R[F]$-flat.

$\implies R[X,Y]$ is an $\mathbb{A}^1$-fibration over $R[F]$
$\implies$ ($\Omega_{R[F]}(R[X,Y]$ is free $R[X,Y]$-module) by Asanuma [Asa87]) $R[X,Y]^{[m]} = R[F]^{[m+1]}$
$\implies$ ($\mathbb{Q} \hookrightarrow R$ or R is seminormal, by Hamann ([Ham75])) $R[X,Y] = R[F]^{[1]}$.

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Residual coordinates of polynomial algebras ([BD93])

Theorem 3 (Bhatwadekar-Dutta, [BD93])

R a Noetherian ring and $W \in R[X, Y]$. Then, the following are equivalent.

1. *W is a residual coord. in $R[X, Y]$.*
2. *$R[X, Y]^{[m]} = R[W]^{[m+1]}$ for some $m \geq 0$.*
3. *$R[W] \subseteq R[X, Y] \subseteq R[W]^{[n]}$ for some $n \in \mathbb{N}$.*

Observation: If W is a residual coord. of $R[X, Y]$, then $R[X, Y]$ is $R[W]$-flat.

Sketch of the proof:

Residual coordinates of polynomial algebras ([BD93])

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Observation: If W is a residual coord. of $R[X,Y]$, then $R[X,Y]$ is $R[W]$-flat.

Sketch of the proof:
(1) $\implies$ (2): Asanuma ([Asa87]).
(2) $\implies$ (3): Trivial.
(3) $\implies$ (1): $R[W]$ is a retract of $R[X,Y]$ and therefore, the inclusions in $R[W] \subseteq R[X,Y] \subseteq R[W]^{[n]}$ are preserved under $\otimes_R k(P)$ for each $P \in \mathrm{Spec}(R) \implies$ $k(P)[W] \hookrightarrow k(P)[X,Y] \hookrightarrow k(P)[W]^{[n]}$ for each $P \in \mathrm{Spec}(R) \implies$ (By Abhyankar-Eakin-Heinzer, [AEH72]) $k(P)[X,Y] = k(P)[W]^{[1]}$ for each $P \in \mathrm{Spec}(R) \implies W$ is a residual coord. of $R[X,Y]$.

Residual coordinates of $R[X, Y]$ are variables! [BD93]

Theorem 4 (Bhatwadekar-Dutta, [BD93])

R a Noetherian ring either containing $\mathbb{Q}$ or R_{red} is seminormal. $W \in R[X, Y]$ is residual coord. (equiv. $R[W] \subseteq R[X, Y] \subseteq R[W]^{[\ell]}$) if and only if $R[X, Y] = R[W]^{[1]}$. (Review GET)

Corollary 5 (Bhatwadekar-Dutta, [BD93])

R a ring and $\underline{W} := (W_1, W_2, \cdots, W_m) \in A := R[X_1, X_2, \cdots, X_n]$ where $m < n$ and $n \geq 2$.

(A) *The following are equivalent if $m = n - 1$:*

1. $A \otimes_R k(P) = (R[\underline{W}] \otimes_R k(P))^{[1]}$.
2. $A^{[\ell]} = R[\underline{W}]^{[1+\ell]}$ *for some* $\ell \geq 0$.
3. $R[\underline{W}] \subseteq A \subseteq R[\underline{W}]^{\ell}$ *for some* $\ell \in \mathbb{N}$.

Further, (1) $\implies$ (2) $\implies$ (3) holds even when $m \neq n - 1$.

(B) *If $m = n - 1$ and either $\mathbb{Q} \hookrightarrow R$ or R_{red} is seminormal, then*

$$R[\underline{W}] \subseteq A \subseteq R[\underline{W}]^{\ell} \text{ for some } \ell \in \mathbb{N} \implies A = R[\underline{W}]^{[1]}.$$

Examples

Example: Theorem 4 does not hold if R does not contain $\mathbb{Q}$ or R_{red} is seminormal.

$R = \mathbb{Z}_{(2)}[2\sqrt{2}]$. If $t = \sqrt{2}$, then $t^2, t^3 \in R$ and $t \notin R$: R is not seminormal.
$W = X - 2Y(tX - Y^2) + t(tX - Y^2)^2 - t(Y - t(tX - Y^2))^4 \in R[X, Y]$. W is a residual coord., but $R[X, Y]/(W) \neq R^{[1]}$ and therefore $R[X, Y] \neq R[W]^{[1]}$.

Question: If W is residual coord. of $R[X, Y]$ and $R[X, Y]/(W) = R^{[1]}$, is then $R[X, Y] = R[W]^{[1]}$?

Example: Theorem 4 does not hold under the setup $R[W] \subseteq R[X, Y, Z](= R^{[3]}) \subseteq R[W]^{[n]}$:

$R = k$, a field.
$X = U^2 + W, \; Y = V^2(U^2 + W) + 2UV + 1, \; Z = V(U^2 + W) + U$
See that $W = XY - Z^2$ and $k[W] \subset k[X, Y, Z] \subset k[U, V, W]$,
but W is not a variable of $R[X, Y, Z]$.

Residual coordinates which is a line but not a variable

Question: R a Noetherian domain and $W \in R[X,Y]$ is a residual coord. of $R[X,Y]$. If $R[X,Y]/(W) = R^{[1]}$, is then W a variable of $R[X,Y]$?

Asanuma-Dutta, On a residual coordinate which is a non-trivial line. J. Pure Appl. Algebra 225 (2021), no. 4, Paper No. 106523, [AD21].

Theorem 6 (Asanuma-Dutta, [AD21])

k an infinite field, $ch(k) = p > 2$, $R := k[[t^2, t^3]]$, $\widetilde{R} := k[[t]]$, the normalisation of R, $I := (t^2, t^3)R = t^2\widetilde{R}$, the conductor ideal of $\widetilde{R}$ in R.

Let $\overline{\tau} : (X,\ Y) \mapsto (X + \overline{t}X^pY^p,\ Y) \in Aut_{\widetilde{R}/C}(\widetilde{R}/C[X,Y])$. Then,

- *There exists $\tau \in Aut_{\widetilde{R}}(\widetilde{R}[X,Y])$ such that τ is a lift of $\overline{\tau}$.*
- $\tau(Y) \in R[X,Y]$
- *$\tau(Y)$ is a residual coordinate of $R[X,Y]$*
- $R[X,Y]/(\tau(Y)) = R^{[1]}$.
- $R[X,Y]/(\tau(Y) - 1) \neq R^{[1]} \implies \tau(Y)$ *is not a coordinate of $R[X,Y]$.*

Applications

Bhatwadekar-Dutta, On residual variables and stably polynomial algebras. Comm. Algebra 21 (1993), no. 2, 635–645, [BD93]:

Theorem 7 (Bhatwadekar-Dutta, [BD93])

R a Noetherian domain, $ch(R) = 0$, either $\mathbb{Q} \hookrightarrow R$ or R is seminormal.
$R[X, Y, Z]/(F) = R[U, V] = R^{[2]}$ and there exists $W \in R[X, Y]$ such that $R[U, V] = R[\bar{W}]^{[1]}$.
Then $R[X, Y, Z] = R[W, F]^{[1]}$.

Proof: If $F \in R[X, Y]$, then $(R[X, Y]/(F))[Z] = R^{[2]}$ and by Hamannn ([Ham75]) and Generalized Epimorphism theorem ([Bha88]) $R[X, Y] = R[F]^{[1]}$ and hence $R[X, Y, Z] = R[F, Z]^{[1]}$.

If $F \notin R[X, Y]$, then $R[W] \subseteq R[X, Y] \hookrightarrow R[X, Y, Z]/(F) = R[U, V] = R[W]^{[1]} \implies$ ([BD93]) W is a residual coord. of $R[X, Y]$ and hence by [BD93] $R[X, Y] = R[W, V]$. Thus, $R[W][V, Z]/(F) = R[W]^{[1]} \implies$ (Generalized Epimorphism theorem, [Bha88]) $R[W][V, Z] = R[W][F]^{[1]}$, i.e., $R[X, Y, Z] = R[W, F]^{[1]}$.

Applications

Das-Dutta, Planes of the form $b(X,Y)Z^n - a(X,Y)$ over a DVR. J. Commut. Algebra 3 (2011), no. 4, 491–509, [DD11]:

Das-Dutta extended an Epimorphism result of Wright over algebraically closed field to any field:

Theorem 8 (Das-Dutta, [DD11])

k a field, $ch(k) = p \geq 0$, $g := b(X,Y)Z^n - a(X,Y) \in k[X,Y,Z]$, $b \neq 0$, $p \nmid n$.

If $B := k[X,Y,Z]/(g) = k^{[2]}$, then exist variables $U, V \in B$ such that V is the image of Z in B, $U \in k[X,Y]$, $b \in k[U]$, $k[X,Y] = k[U,a]$ and $k[X,Y,Z] = k[U,g,Z]$.

Using the theory of residual coord. the above result gets generalized to the ring case.

Theorem 9 (Das-Dutta, [DD11])

k a field and $k \hookrightarrow R$ a Noetherian domain. $ch(k) = p \geq 0$.
$g := b(X,Y)Z^n - a(X,Y) \in R[X,Y,Z]$, $b \neq 0$, $p \nmid n$.

If $R[X,Y,Z]/(g) = R^{[2]}$, then $R[X,Y] = R[a]^{[1]}$ and $R[X,Y,Z] = R[a,Z]^{[1]} = R[g,Z]^{[1]}$ provided either $\mathbb{Q} \hookrightarrow R$ or if R is seminormal.

Applications

Berson-Bikker-van den Essen: Adapting coordinates. J. Pure Appl. Algebra 184 (2003), no. 2-3, 165–174, [BBE03]:

Corollary 10 (Berson-Bikker-van den Essen, [BBE03])

R a ring. $F \in R[X, Y]$ and $a \in R$.

F is a coordinate of $R[X, Y]$ if and only if it remains coordinate in $R_a[X, Y]$ and in $R/aR[X, Y]$.

Conjecture: R a ring and $a \in R$. If $(F_1, F_2, \cdots, F_{n-1}) \in R[X_1, X_2, \cdots, X_n]$ is a partial coordinate system of $R_a[X_1, X_2, \cdots, X_n]$ and of $R/aR[X_1, X_2, \cdots, X_n]$, then $(F_1, F_2, \cdots, F_{n-1})$ is a partial coordinate-system of $R[X_1, X_2, \cdots, X_n]$.

In [BBE03]: shown that if a is a non-zero divisor, then the conjecture is true.

Lahiri, A note on partial coordinate system in a polynomial ring. Comm. Algebra 47 (2019), no. 3, 1099–1101, **[Lah19]: The conjecture holds true; proof uses theory of residual coordinates.**

Applications: New tools

Bhatwadekar-Dutta, Kernel of locally nilpotent R-derivations of $R[X, Y]$. Trans. Amer. Math. Soc. 349 (1997), no. 8, 3303–3319 [BD97]:

Theorem 11 (Bhatwadekar-Dutta, [BD97])

$\mathbb{Q} \hookrightarrow R$ a Noetherian domain with $Qt(R) = K$. $F \in R[X, Y]$ be a such that $K[X, Y] = K[F]^{[1]}$.

Then $R[X, Y] = R[F]^{[1]}$ if and only if $(F_X, F_Y)R[X, Y] = R[X, Y]$.

Proof: Technique: Residual coordinates. To show: if $(F_X, F_Y)R[X, Y] = R[X, Y]$, then F is a residual coord.. Note: If F is a residual coord. of $R_P[X, Y]$ for all $P \in \mathrm{Spec}(R)$, then F is a residual coord. of $R[X, Y]$. Assume R is local $\implies \dim(R) < \infty$. Rest of the proof uses induction on $\dim(R)$, mainly on dim 0 and dim 1.

Theorem 12 (Bhatwadelar-Dutta, [BD97], Daigle-Freudenburg Noetherian UFD-case, [DF98])

$\mathbb{Q} \hookrightarrow R$ a Noetherian domain

If $D \in LND_R(R[X, Y])$ is fixed point free, then D has a slice.

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Residual coordinates of affine fibrations, [EK13]

Question: R a polynomial ring over a field $k \hookleftarrow \mathbb{Q}$. Are residual coordinates or stable coordinates of $R^{[3]}$ are coordinates? (Open) Are residual coordinates and stable coordinates of $R^{[3]}$ are the same? (Yes)

Definition 13 (Kahoui, [EK13])

R a ring, A an $\mathbb{A}^n$-fibration over R and $W \in A$.
W is called a residual coord. of A if $A \otimes_R k(P) = (R[W] \otimes_R k(P))^{[n-1]}$ for all $P \in \operatorname{Spec}(R)$.

Theorem 14 (Kahoui, [EK13])

$R = \mathbb{C}^{[n]}$ *for some* $n \geq 0$, A *be an* $\mathbb{A}^3$*-fibration over* R *and* $W \in A$.

Then W *is a residual coordinate of* A *iff* W *is a stable coordinate of* A *iff* A *is an* $\mathbb{A}^2$*-fibration over* $R[W]$.

Question: If R is any ring containing $\mathbb{Q}$, whether residual coordinates of $\mathbb{A}^3$-fibrations are stable coordinates.

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Residual coordinates of affine fibrations, [DD14]

Definition 15 (Das-Dutta, [DD14])

R a ring, A an R-algebra, $n \in \mathbb{N}$, $\underline{W} := (W_1, W_2, \cdots, W_m)$ an m-tuple of elements in A which are algebraically independent over R such that $A \otimes_R k(P) = (R[\underline{W}] \otimes_R k(P))^{[n-m]}$ for all $P \in Spec(R)$. We shall call such an m-tuple $\underline{W}$ to be an m-tuple residual coord. of A over R.

Their observations:

- If A is an $\mathbb{A}^n$-fibration over R and $\underline{W}$ a m-tuple residual coord. of A, then A is an $\mathbb{A}^{n-m}$-fibration over $R[\underline{W}]$. Further, $\Omega_R(A)$ is a stably free A-module if and only if $\Omega_{R[\underline{W}]}(A)$ is a stably free A-module.
- R a Noetherian ring and A an $\mathbb{A}^{m+1}$-fibration over R. Then $\underline{W}$ is an m-tuple residual coord. of A over R iff A is an $\mathbb{A}^1$-fibration over $R[\underline{W}]$. Consequently, if R is a Noetherian UFD, then $\underline{W}$ is an m-tuple residual coord. of A over R if and only if $A = R[\underline{W}]^{[1]} = R^{[m+1]}$.
- R a finite-dimensional polynomial algebra over a PID, A an $\mathbb{A}^n$-fibration over R and $\underline{W}$ an m-tuple residual coord. of A over R. Then A is a stably polynomial algebra over $R[\underline{W}]$.

Recall ...

- $W \in R[X,Y]$ is called a residual coordinate if $R[X,Y] \otimes_R k(P) = (R[W] \otimes_R k(P))^{[1]} = k(P)[\overline{W}]^{[1]}$ for each $P \in \mathrm{Spec}(R)$.
- W is a residual coord. of $R[X,Y]$
  1. $\implies R[X,Y]$ is an $\mathbb{A}^1$-fibration over $R[W]$.
  2. $\Leftrightarrow R[X,Y]^{[m]} = R[W]^{[n]} \Leftrightarrow R[W] \subseteq R[X,Y] \subseteq R[W]^{[n]}$.
- If either $\mathbb{Q} \hookrightarrow R$ or R_{red} is seminormal, then W is a residual coord $\Leftrightarrow R[X,Y] = R[W]^{[1]}$.
- $R = \mathbb{Q}[t]_{(t)}$, $F = t^2Y + (X + tY^2) + t(X + tY^2)^2$ is a residual coord of $R[X,Y] \implies R[X,Y] = R[F]^{[1]}$.
- $\underline{W} := (W_1, W_2, \cdots, W_m) \in R[X_1, X_2, \cdots, X_n]$ residual coord. tuple
  1. $\implies R[X_1, X_2, \cdots, X_n]^{[\ell]} = R[\underline{W}]^{[\ell+n-m]}$.
  2. $\Leftrightarrow R[X_1, X_2, \cdots, X_n]^{[\ell]} = R[\underline{W}]^{[\ell+1]}$, if $n - m = 1$.
- If $n - m = 1$ and either $\mathbb{Q} \hookrightarrow R$ or R_{red} is seminormal then $\underline{W}$ is a residual coord tuple $\Leftrightarrow R[X_1, X_2, \cdots, X_n] = R[\underline{W}]^{[1]}$.
- $\mathbb{Q} \hookrightarrow R$ a Noetherian domain with $\mathrm{Qt}(R) = K$. $F \in R[X,Y]$ be a such that $K[X,Y] = K[F]^{[1]}$. Then $R[X,Y] = R[F]^{[1]}$ if and only if $(F_X, F_Y)R[X,Y] = R[X,Y]$.

Recall ...

- If $\underline{W} := (W_1, W_2, \cdots, W_m)$ an m-tuple of elements in A which are algebraically independent over R such that $A \otimes_R k(P) = (R[\underline{W}] \otimes_R k(P))^{[n-m]}$ for all $P \in Spec(R)$, then $\underline{W}$ is called an m-tuple residual coord. of A over R.

  Let A is an $\mathbb{A}^n$-fibration over R and $\underline{W}$ a m-tuple residual coord. of A, then
- A is an $\mathbb{A}^{n-m}$-fibration over $R[\underline{W}]$.
- $\Omega_R(A)$ is a stably free A-module if and only if $\Omega_{R[\underline{W}]}(A)$ is a stably free A-module.
- Let $n - m = 1$ and R a Noetherian ring. Then $\underline{W}$ is an m-tuple residual coord. of A over R iff A is an $\mathbb{A}^1$-fibration over $R[\underline{W}]$.

Residual coordinates of affine fibrations

Theorem 16 (Das-Dutta, [DD14])

R a Noetherian ring and A an $\mathbb{A}^n$-fibration over R such that $\Omega_R(A)$ is a stably free A-module.

If $\underline{W}$ is an m-tuple residual coord. of A over R, $A^{[\ell]} = R[\underline{W}]^{[n-m+\ell]}$ for some $\ell \in \mathbb{N}$.

Further, if $m = n-1$ and either $\mathbb{Q} \hookrightarrow R$ or R_{red} is seminormal, then $\underline{W}$ is a residual coordinate of A if and only if $A = R[\underline{W}]^{[n-1]}$.

Example of Hochster (Res. cord is stable coord. and not a coord.)

$R := \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1) = R[x, y, z]$. Define and R-LND on $R[U, V, W]$ by $D_0(U) = x$, $D_0(V) = y$ and $D_0(W) = z$.

Check that $D_0(xU + yV + zW) = 1$, and therefore, by Slice Theorem we have $R[U, V, W] = A[xU + yV + zW] = A^{[1]}$ where $A := \text{Ker}(D_0)$. Let $s = xU + yV + zW$ $R[s] \subset R[U, V, W] = A[s]$. Check that s is a residual coord. of $R[U, V, W] \implies s$ is a stable coord. of $R[U, V, W]$. However, it is not a coord. since $A \neq R^{[2]}$.

Application: A cancellation problem

Question: R a ring, A an R-algebra and $A[T] = R[U, V, W]$. Is then $A = R^{[2]}$?
A possible approach: Find $F \in A[T] \setminus A$ such that $A[T] = R[U, V, W] = R[F]^{[2]}$. Identify A as a subring of $A[T]/(F) = R^{[2]}$. Then one explicitly constructs variables for A in terms of judiciously chosen variables for $A[T]/(F)$ exploiting the fact that $A[T]/(F)$ is a simple ring extension of A.

Such approach was taken by Sathaye ([Sat76]) and Russell ([Rus76] for the case $F = bT - a$ and Wright [Wri78] for the case $F = bT^n - a$, $n \geq 2$. ([RS79])

Theorem 17 (Wright ([Wri78]), Das ([Das15]))

k a field, $ch(k) = p \geq 0$, A a normal affine k-domain. $a, b \in A$, $b \neq 0$, $A[T]/(bT^n - a) = k^{[2]}$, $n \geq 2$, $p \nmid n$

Then, there exist variables X, Y in $A[T]/(bT^n - a)$ such that Y is the image of T in $A[T]/(bT^n - a)$, $b \in k[X]$, $A = k[X, a] = k^{[2]}$ and $A[T] = k[X, bT^n - a, T] = k^{[3]}$.

Question: Does it hold over domains?

Theorem 18 (Das, [Das15])

k a field, $ch(k) = p \geq 0$, $k \hookrightarrow R$ a Noetherian normal domain, A a finitely generated flat R-domain with $\Omega_R(A)$ stably free A-module, $a, b \in A$ such that $A[T]/(bT^n - a) = R^{[2]}$, $n \geq 2$ and $p \nmid n$. Suppose, for each $P \in Spec(R)$, we have $A \otimes_R k(P)$ is normal and $b \nmid PA_P$.

Then, $A = R[a]^{[1]} = R^{[2]}$ and $A[T] = R[bT^n - a, T]^{[1]} = R^{[3]}$. When R is UFD, the hypothesis "$\Omega_R(A)$ stably free" may be dropped.

Corollary 19 (Das, [Das15])

k a field, $ch(k) = p \geq 0$, $k \hookrightarrow R$ a Noetherian normal domain, A and R-algebra such that $A[T] = R[bT^n - a]^{[2]} = R^{[3]}$ where $n \geq 2$ and $p \nmid n$

Then, $A = R[a]^{[1]} = R^{[2]}$ and $A[T] = R[bT^n - a, T]^{[1]}$.

Corollary 20 (Das, [Das15])

$\mathbb{Q} \hookrightarrow R$ a Noetherian UFD, A an $\mathbb{A}^2$-fibration over R, $a, b \in A$, $n \geq 2$ such that $A[T]/(bT^n - a) \otimes_R k(P) = k(P)^{[2]}$ for all $P \in Spec(R)$

Then, $A = R[a]^{[1]} = R^{[2]}$ and $A[T] = R[bT^n - a, T]^{[1]}$.

Application: Another cancellation problem

Question: Over any one dimensional domain $R \hookleftarrow \mathbb{Q}$, is a $\mathbb{A}^2$-fibration A a polynomial algebra?
Comment in [AB97]: "$\Omega_R(A)$ being not free is the only obstruction for the result of Sathaye to be not true for an arbitrary local domain R of dimension 1".

Improvement: If A is an $\mathbb{A}^2$-fibration over a one dimensional Noetherian domain $R \hookleftarrow \mathbb{Q}$ such that $\Omega_R(A)$ is stably free A-module, then $A = R^{[2]}$.
Proof: A is an $\mathbb{A}^1$-fibration over a one dimensional Noetherian domain $R \hookleftarrow \mathbb{Q} \implies$ ([AB97]) A is an $\mathbb{A}^1$-fibration over $R[W]$ for some $W \in A \implies$ ([DD14]) W is a residual coord. of A $\implies$ ($\Omega_R(A)$ stably free A-module, [DD14]) $A = R[W]^{[1]} = R^{[2]}$.

Theorem 21 (Kahoui-Ouali, [EKO14])

$\mathbb{Q} \hookrightarrow R$ Noetherian one-dimensional domain and A an R-algebra.

If $A^{[n]} = R^{[n+2]}$, then $A = R^{[2]}$.

Proof:
$\mathbb{Q} \hookrightarrow R$ Noetherian and $A^{[n]} = R^{[n+2]} \implies A$ is an $\mathbb{A}^2$-fibration over $R \implies$ (by previous result) $A = R^{[2]}$.

Application: Fixed point free LND of $\mathbb{A}^2$-fibration

Known: If $\mathbb{Q} \hookrightarrow R$ is a ring and D is a fixed point free R-LND of $R[X, Y]$, then D has a slice, i.e., $R[X, Y] = \text{Ker}(D)^{[1]}$, and in that case $\text{Ker}(D) = R^{[1]}$ ([Ren68], [DF98], [BD97], [BvM01], [Ess07]). **Question:** What happens when $R[X, Y]$ is replaced by an $\mathbb{A}^2$-fibration?

Theorem 22 (Kahoui-Ouali, [EKO16])

$\mathbb{Q} \hookrightarrow R$ a ring and A an R-algebra such that $A^{[m]} = R^{[m+2]}$. If D is a fixed point free R-LND of A, then D has a slice, i.e., $A = Ker(D)^{[1]}$ and in that case $Ker(D) = R^{[1]}$ (i.e., $A = R^{[2]}$).

Proof:(Assume R domain; $\text{Qt}(R) = K$) By a reduction method assume R is a Notherian f.g. $\mathbb{Q}$-domain. Consider D_K on $A \otimes_R K = K^{[2]}$. $\text{Ker}(D_K) = K[U_1]$ for some $U_1 \in A$. Extend D trivially to $\widetilde{D}$ on $A^{[m]} = A[\underline{T}] = R[\underline{X}] = R^{[m+2]}$. Compare with $\widetilde{D}$ with $\mathcal{JD}_{(\underline{X})}(U_1, \underline{T}, -)$ to see $\widetilde{D} = \mathcal{JD}_{(\underline{X})}(U, \underline{T}, -)$ where $aU = U_1 + r$ where $r \in R$ and $a \in A$. Now show that U is actually a residual coord. of $A \implies (A^{[m]} = R^{[m+2]}$, [DD14]) $A = R[U]^{[1]} = R^{[2]} \implies D$ has a slice.

Babu-Das, [BD21]: $\mathbb{Q} \hookrightarrow R$ a Noetherian ring, A an $\mathbb{A}^1$-fibration over R. If D is a fixed point free R-LND of A, then D has a slice, i.e., $A = \text{Ker}(D)^{[1]}$ and in that case $\text{Ker}(D)$ is an $\mathbb{A}^1$-fibration.

Application: New tool

Theorem 23 (Babu-Das, communicated)

$\mathbb{Q} \hookrightarrow R$ a Noetherian domain, $Qt(R) = K$ and A an R-algebra such that A is a retraction of $B = R[X_1, X_2, \cdots, X_n] = R^{[n]}$ and $tr.deg_R(A) = 2$.

If $F \in A$ be such that $A \otimes_R K = K[F]^{[1]}$ and $(F_{X_1}, F_{X_2}, \cdots, F_{X_n})B = B$, then F is a residual coordinate of A.

Corollary 24 (Babu-Das, communicated)

$\mathbb{Q} \hookrightarrow R$ a Noetherian domain, $Qt(R) = K$, A an R-algebra such that $A^{[n]} = A[\underline{T}] = R[\underline{X}] = R^{[n+2]}$ where $\underline{X} = (X_1, X_2, \cdots, X_{n+2})$ and $\underline{T} = (T_1, T_2, \cdots, T_n)$. Let $F \in A$ be such that $A \otimes_R K = K[F]^{[1]}$. Then, TFAE:.

(I) $(F_{X_1}, F_{X_2}, \cdots, F_{X_{n+2}})A[\underline{T}] = A[\underline{T}]$.

(II) *F is a residual coordinate of A.*

(III) $A = R[F]^{[1]} = R^{[2]}$.

(IV) $\mathcal{JD}_{(\underline{X})}(F, \underline{T}, -)$ *is a fixed point free R-LND.*

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Residual coordinates of R^n are m-stable coordinates: Bounds on m

[BD93], [DD14]: Any residual coordinate of R^n is a m-stable coordinate for some $m \in \mathbb{N}$.

Theorem 25 (Kahoui-Ouali, [EKO18])

(I): $\mathbb{Q} \hookrightarrow k$ *algebraically closed field,* $R = k[X]$*. Then, any residual coordinate of* $A = R^{[n]}$*, where* $n \geq 3$*, is a* 1*-stable coordinate.*

(II): $\mathbb{Q} \hookrightarrow R$ *a Noetherian d-dimensional ring. Then, any residual coordinate of* $A = R^{[n]}$ *is a* $((2^d - 1)n)$*-stable coordinate.*

Theorem 26 (Dutta-Lahiri, [DL21])

(I): *k algebraically closed field, R one-dimensional affine k-algebra, either* $ch(k) = 0$ *or or* R_{red} *is seminormal. Then, any residual coordinate of* $R^{[n]}$*,* $n \geq 3$*, is a* 1*-stable coordinate.*

(II): *If R is a Noetherian d-dimensional ring, then every residual coordinate of* $R^{[n]}$ *is a* $((2^d - 1)n)$*-stable coordinate.*

(III): *k algebraically closed field,* $ch(k) = 0$*, R a f.g. k- algebra,* $\dim(R) = d$*. Then every residual cooord. of* $R^{[n]}$ *is* $(2^d - 1)n - 2^{d-1}(n-1) = (2^{d-1}(n+1) - n)$*-stable coordinate.*

Residual coordinates of R^n are m-stable coordinates: Bounds on m

Theorem 27 (Kahoui-Essamaoui-Ouali, [EKEO21])

R a Noetherian, $\dim(R) = 1$. Then the following holds.

1. *Every residual coord. of $A = R^{[2]}$ is a 1-stable coordinate.*
2. *If R is an integral ring extension of $k^{[1]}$, where k is an algebraically closed field, then for every $n \geq 3$, residual coordinates of $R^{[n]}$ are 1-stable coordinates.*
3. *If R is a complete local ring containing a field then for every $n \geq 3$, residual coordinates of $R^{[n]}$ are 1-stable coordinates.*

Example of Bhatwadelar-Dutta

Example 28 (Bhatwadekar-Dutta, [BD94], Vénéreau (2001, thesis))

$\mathbb{Q} \hookrightarrow k$ be a field, $R = k[\pi]_{(\pi)}$ and $A = R[X, Y, Z]$. Set $F := \pi^2 X + \pi Y(YZ + X + X^2) + Y$. One can check that $A \otimes_R k(P) = (R[F] \otimes_R k(P))^{[2]}$ for all $P \in \operatorname{Spec}(R) \implies$ F is a residual coord. of $A \implies A^{[m]} = R[F]^{[m+2]}$, in fact, it can be shown that $A^{[1]} = R[F]^{[3]}$ (also follows from [EKEO21])

It is not known whether $A = R[F]^{[2]}$.

Define an R-LND D of A by $D(X) = Y^2$, $D(Y) = 0$ and $D(Z) = -(\pi + Y + 2XY)$. Then, $R[F] \subseteq \operatorname{Ker}(D)$. It is known that $\operatorname{Ker}(D) = R[F]^{[1]} = R^{[2]}$. We now show that D is not fixed point free.

On the contrary, assume that D is fixed point free, and therefore, there exists $f_1, f_2, f_3 \in R[X, Y, Z]$ such that $D(X)f_1 + D(Y)f_2 + D(Z)f_3 = 1$.
Since $D(Y) = 0$, we have $D(X)f_1 + D(Z)f_3 = 1$, i.e., $Y^2 f_1 - (\pi + Y + 2XY)f_3 = 1$.
Hence, in $A/YA = R[X, Z]$ we get $-\pi f_3 = 1$, i.e., π is a unit in $R[X, Z]$ – a contradiction to the fact that π is a prime in R.

Thank you!

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